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EXACT RESULTS FOR THE SUPERSYMMETRIC G_2 GAUGE THEORIES

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Abstract

We study the $N = 1$ supersymmetric gauge theories with N_f flavors of quarks in the fundamental vector representation. We find dynamically generated superpotentials, smooth quantum moduli space, quantum moduli space with additional mesons, non trivial IR fixed points.

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1 Introduction and conclusion.

In the recent period there has been a lot of interest in supersymmetric gauge theories; this is due to the fact that certain quantities are often exactly calculable because they are extremely constrained by holomorphy and symmetries (for a review see ref. ([1])). These exact results provide insight in the strong coupling region where a lot of new interesting phenomena take place. Some of them are universal, other specific.

One feature, which seems to be universal, is the sequence of phases into which the theory with massless quarks moves when the number of flavors is increased (for a short review see refs ([2])). First when there are few flavors the theory has a dynamically generated superpotential without ground state, then it moves into a confining phase which is eventually followed by a free magnetic one. Increasing further the number of massless quarks the theory changes phase to a non-Abelian Coulomb phase and finally it stops being asymptotically free and therefore it moves into the free electric phase.

A second feature, which is perhaps the most exciting one, is the existence of a dual description in terms of a magnetic theory in the deep IR for a certain range of the parameter N_f , the number of flavors. In almost all the examples so far known the dual theory has a different gauge group, which nevertheless belongs to the same series of the Lie classification of the original one ([5], [6], [7], [8]). An exception to this pattern is represented by the $SU(2k)$ theory with an antisymmetric tensor and vector matter ([9]) whose dual is the product of two groups. Nevertheless in all the known cases it is true that the dual of a simple group with matter in the elementary vector representation is in the same Lie series.

It is therefore interesting trying to understand whether these features are preserved in the case of special groups. In particular it would be very interesting to know which is the dual of the special groups. The purpose of this work is to examine the simplest of all the special groups, i.e. G_2 . We find that at least the first feature is preserved while we have not being able to show whether the second is maintained.

A summary of our results is as follows: for $N_f \leq 3$ flavours of vector matter we find dynamically generated superpotentials associated either with gaugino condensation for $N_f < 3$ or with instantons for $N_f = 3$ which lift the classical vacuum degeneracy and imply the non existence of a vacuum. For $N_f > 3$ there is still a quantum moduli space. In a generic point of

which the group is completely broken by the Higgs mechanism. Classically there are singular submanifold where there is an enhanced symmetry because some W bosons are massless on them. Quantistically for $N_f = 4$ the moduli space is completely smooth, without any singularity, because of instanton effect: the origin does not belong to it. The theory confines. For $N_f = 5$ the quantum moduli space is the same of the classical one but the singularity at the origin implies the existence of new mesons rather than massless W ; again at the origin the theory confines. For $6 \leq N_f < 12$ the theory flows to a non trivial interacting superconformal field theory, which implies a non-Abelian Coulomb phase.

2 The classical moduli space

We will study the $N = 1$ supersymmetric QCD with gauge group G_2 and N_f flavours of quarks Q^{cf} in the (real) fundamental $\underline{7}$ representation ($c = 1 \dots 7$, $f = 1 \dots N_f$). The Wilsonian (one loop) beta function is

$$\beta_W = -\frac{g^3}{16\pi^3}(12 - N_f) \quad (2.1)$$

and therefore the theory is asymptotically free for $N_f < 12$. The global symmetries are

	$SU(N_f)$	$U(1)_R$	$U(1)_{Q^{f_0}}$
Q^f	N_f	$\frac{N_f-4}{N_f}$	δ_{ff_0}
W	1	1	0
$\Lambda_{N_f}^{12-N_f}$	1	0	2

(2.2)

where the last $U(1)_{Q^{f_0}}$ is anomalous. The theory also has a discrete Z_{2N_f} that acts as $Q \rightarrow e^{\frac{i\pi}{2N_f}} Q$ which is the part of $U(1)_A$ left unbroken by the anomaly.

Also in the case of G_2 it is possible to obtain the explicit form of the classical moduli space; using the explicit realization of the G_2 generators

given in the appendix we find (all the parameters are real)

$$\| Q_c^f \| = \begin{pmatrix} b_3 e^{\frac{i}{2}(2\phi_{16}+\phi_{31}-\phi_{34}+\pi)} & 0 & a_3 e^{i\phi_{31}} & 0 & 0 & \dots & 0 \\ b_4 e^{i(\phi_{16}+\frac{\phi_{42}}{2}-\frac{\phi_{43}}{2})} & 0 & 0 & a_4 e^{i\phi_{42}} & 0 & \dots & 0 \\ b_4 e^{i(\phi_{16}-\frac{\phi_{42}}{2}+\frac{\phi_{43}}{2}+\pi)} & 0 & 0 & a_4 e^{i\phi_{43}} & 0 & \dots & 0 \\ b_3 e^{\frac{i}{2}(2\phi_{16}-\phi_{31}+\phi_{34}+3\pi)} & 0 & a_3 e^{i\phi_{34}} & 0 & 0 & \dots & 0 \\ b_2 e^{i(\phi_{16}+\frac{\phi_{25}}{2}-\frac{\phi_{27}}{2})} & a_2 e^{i\phi_{25}} & 0 & 0 & 0 & \dots & 0 \\ a_1 e^{i\phi_{16}} & 0 & 0 & 0 & 0 & \dots & 0 \\ b_2 e^{i(\phi_{16}-\frac{\phi_{25}}{2}+\frac{\phi_{27}}{2}+\pi)} & a_2 e^{i\phi_{27}} & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (2.3)$$

along these flat directions there is the following pattern of symmetry breaking $G_2 \xrightarrow{a_1 \neq 0} SU(3) \xrightarrow{a_2 \neq 0} SU(2) \xrightarrow{a_3 \neq 0} \emptyset$ and the generators left unbroken are $\{H, E_\alpha\} \xrightarrow{a_1 \neq 0} \{H_1, H_2, E_{\pm 2}, E_{\pm 4}, E_{\pm 6}\} \xrightarrow{a_2 \neq 0} \{H_2, E_{\pm 4}\}$. It turns out examining the gauge invariant description of the moduli space eq.s (2.7) that the sequence in which we switch on the v.e.v is $a_1, a_2, a_3, b_4, a_4, b_1, b_2, b_3$.

We can now write down the gauge invariant operators using the two invariant tensors g_{cd} and $\Gamma_{c_1 c_2 c_3}$

$$\begin{aligned} M^{fg} &= M^{gf} = g_{cd} Q^{cf} Q^{dg} \\ B^{f_1 f_2 f_3} &= \Gamma_{c_1 c_2 c_3} Q^{c_1 f_1} Q^{c_2 f_2} Q^{c_3 f_3} \quad N_f \geq 3 \\ \mathcal{B}^{f_1 f_2 f_3 f_4} &= \Gamma_{[c_1 c_2}^p \Gamma_{c_3 c_4]p} Q^{c_1 f_1} Q^{c_2 f_2} Q^{c_3 f_3} Q^{c_4 f_4} \quad N_f \geq 4 \end{aligned} \quad (2.4)$$

with their charges under the global symmetries

	$SU(N_f)$	$U(1)_R$	$U(1)_{Q^{f_0}}$
M^{fg}	$\frac{N_f(N_f+1)}{2}$	$2\frac{N_f-4}{N_f}$	$\delta_{ff_0} + \delta_{gf_0}$
$B^{f_1 f_2 f_3}$	$\binom{N_f}{3}$	$3\frac{N_f-4}{N_f}$	$\sum_i \delta_{f_i f_0}$
$\mathcal{B}^{f_1 \dots f_4}$	$\binom{N_f}{4}$	$4\frac{N_f-4}{N_f}$	1

(2.5)

and then use the Bose statistic of Q^{cf} and the proprieties of Γ to deduce relation among these objects.

In particular the fact that we can only use up to two Γ follows from the reducibility of the product of three or more Γ (as shown in appendix eq. (

A.3) and the explicit antisymmetrization over the flavour indices in the last equation of eq. (2.4), which implies the constraint $N_f \geq 4$, is a consequence of

$$\mathcal{B}^{f_1 f_2 | f_3 f_4} = \mathcal{B}^{[f_1 f_2 f_3 f_4]} + 2M^{f_1 [f_3} M^{f_4] f_2} \quad (2.6)$$

The gauge invariant description of the moduli space (2.3) is given by

$$\begin{aligned} M &= \begin{pmatrix} -2a_4^2 e^{i(\phi_{42}+\phi_{43})} & 0 & 0 & 0 \\ 0 & -2a_2^2 e^{i(\phi_{25}+\phi_{27})} & 0 & 0 \\ 0 & 0 & 2a_3^2 e^{i(\phi_{31}+\phi_{34})} & 0 \\ 0 & 0 & 0 & e^{2i\phi_{16}} (a_1^2 + 2\sum_i b_i^2) \end{pmatrix} \\ B^{123} &= i\sqrt{2}a_2a_3b_4e^{\frac{-i}{2}(-2\phi_{16}+\phi_{42}+\phi_{43})} \left(e^{i(\phi_{25}+\phi_{34}+\phi_{42})} + e^{i(\phi_{27}+\phi_{31}+\phi_{43})} \right) \\ B^{124} &= -\sqrt{2}a_2a_4b_3e^{\frac{-i}{2}(-2\phi_{16}+\phi_{31}+\phi_{34})} \left(e^{i(\phi_{25}+\phi_{34}+\phi_{42})} + e^{i(\phi_{27}+\phi_{31}+\phi_{43})} \right) \\ B^{134} &= i\sqrt{2}a_3a_4b_2e^{\frac{-i}{2}(-2\phi_{16}+\phi_{25}+\phi_{27})} \left(e^{i(\phi_{25}+\phi_{34}+\phi_{42})} + e^{i(\phi_{27}+\phi_{31}+\phi_{43})} \right) \\ B^{234} &= i\sqrt{2}a_2a_3a_4 \left(e^{i(\phi_{25}+\phi_{34}+\phi_{42})} - e^{i(\phi_{27}+\phi_{31}+\phi_{43})} \right) \\ \mathcal{B}^{1234} &= \sqrt{2}a_1a_2a_3a_4e^{i\phi_{16}} \left(e^{i(\phi_{25}+\phi_{34}+\phi_{42})} + e^{i(\phi_{27}+\phi_{31}+\phi_{43})} \right) \end{aligned} \quad (2.7)$$

3 $N_f = 1, 2$. A dynamically generated superpotential by gaugino condensation with no vacuum.

Using the symmetries eq. (2.2) the only dynamical generated superpotential, which is allowed, is

$$W_{N_f} = A_{N_f} \left(\frac{\Lambda_{N_f}^{12-N_f}}{\det M} \right)^{\frac{1}{4-N_f}} \quad (3.1)$$

Adding $W_{\text{tree}} = mM$ and integrating away the massive quarks we find $A_{N_f} = (4 - N_f) \left(\frac{A_0}{4} \right)^{\frac{4}{4-N_f}}$. We fix $A_0 = 4$ in such a way that the pure G_2 YM theory has the effective superpotential $W_0 = 4(\Lambda_{N_f}^{12-N_f} \det m)^{\frac{1}{4}}$ which shows that the pure G_2 SYM has 4 different vacua.

As in ref. ([10]) this superpotential is generated by gaugino condensation in the $SU(4 - N_f)$ YM theory left unbroken by $\langle Q \rangle$. This can be checked

along the flat direction with $\langle Q^{c1} \rangle = \delta^{c6} a_1$ of the $N_f = 2$ theory where the low energy theory left by the Higgs mechanism is $SU(3)$ with $N_f = 1$ flavour: we find that the two scales are relate by $\Lambda_{3,1}^8 = \frac{\Lambda_2^{10}}{2a_1^2}$ since $M^{22} = 2\hat{M}^{2|2}$ where $\hat{M}^{2|2}$ is the gauge invariant meson of the $SU(3)$ theory. In the $SU(3)$ SQCD obtained higgsing G_2 the $\underline{\bar{3}}$ quarks transform naturally in the N_f representation of the global gauge group $SU(N_f) \times SU(N_f) \times U(1)_V \times U(1)_R$ exactly as the $\underline{3}$ do but differently from the usual assignment.

4 $N_f = 3$. An instanton generated superpotential with no vacuum.

The superpotential eq. (3.1) also makes sense for $N_f = 3$ and it reduces nicely to W_2 when adding $W_{\text{tree}} = m_{33}M^{33}$ and integrating the third massive quark away but when $N_f = 3$ there is another gauge invariant available besides M^{fg} (eq. (2.4)) and it is $B \equiv B^{123}$

The classical vacua can be described with these 7 gauge invariant operators without any constraints as the naive counting of the real d.o.f shows: $\sharp(Q) - 2\dim(G_2) = \sharp(M) + \sharp(B) = 14$. This can also directly be confirmed with the tensor analysis and by the direct inspection of the classical moduli space (2.7).

Using the symmetries eq. (2.2) we can write the most general superpotential as

$$W_3 = \frac{\Lambda_3^9}{\det M} f\left(\frac{B^2}{\det M}\right) \quad (4.1)$$

where $f(u)$ is an arbitrary function. Adding $W_{\text{tree}} = mM + bB$ we get

$$W_{\text{eff}} = \frac{\Lambda_3^9}{\det M} \bar{f}\left(\frac{B^2}{\det M}, \frac{\det m(\det M)}{(mM)^3}, \frac{mM \det M}{\Lambda_3^9}, \frac{bB \det M}{\Lambda_3^9}\right) \quad (4.2)$$

Since for $\det M \neq 0$ the group is completely broken we expect instanton corrections which contribute as $(\Lambda_3^9)^n$ with $n \geq 0$, i.e. $\bar{f}(x_1, x_2, x_3, x_4) = \sum_{n+m \geq -1} \bar{f}_{nm}(x_1, x_2) x_3^n x_4^m$. The further necessities to have a smooth $m = 0$ and $b = 0$ limit imply

$$W_{\text{eff}} = \frac{\Lambda_3^9}{\det M} f\left(\frac{B^2}{\det M}\right) + mM + bB \quad (4.3)$$

In order to determine f we use the integrate in technique ([3]) (whose notation we use) since the Principle of Linearity is satisfied. Integrating away Q^{c3} at the classical level in W_{tree} we get $W_{\text{tree, d}} = -\frac{b^2}{4m_{33}} \det \hat{M}$ where \hat{M} are the gauge invariant mesons describing the $N_f = 2$ down theory. The most general W_Δ is $W_\Delta = \left(\frac{\Lambda_2^{10}}{\det \hat{M}}\right)^{\frac{1}{2}} w \left(\frac{b^2(\det \hat{M})^3}{\Lambda_2^{10} m_{33}^2}\right)$. As explained in ref. ([3]) we must have $W_\Delta|_{b=0} = 0$, $\lim_{m_{33} \rightarrow \infty} W_\Delta = 0$ and $\lim_{\Lambda_2^{10} \rightarrow 0} W_\Delta = 0$ (because switching off the gauge interaction we are left only with $W_{\text{tree, d}}$) Tuning the limits $m_{33} \rightarrow \infty$ and $\Lambda_2^{10} \rightarrow 0$ we get $W_\Delta = 0$. Assuming simple thresholds, we can now integrate away b and m_{33} from the expression $W_n = \left(\frac{\Lambda_2^{10}}{\det \hat{M}}\right)^{\frac{1}{2}} - \frac{b^2}{4m_{33}} \det \hat{M} - m_{33} M^{33} - bB$. We obtain

$$W_3 = \frac{\Lambda_3^9}{\det M - B^2} \quad (4.4)$$

where $f(u) = \frac{1}{1-u}$. An independent test of this superpotential can be obtained higgsing the theory to $SU(3)$ by $\langle Q^{c1} \rangle = \delta^{c6} a_1$. We find $(f, g \geq 2)$ $M^{fg} = \hat{M}^{f|g} + \hat{M}^{g|f}$ and $B = ia_1(\hat{M}^{2|3} - \hat{M}^{3|2})$ which gives the usual superpotential for $SU(3)$ with $N_f = 2$ when we identify the two scales as $\Lambda_{3,2}^7 = \frac{\Lambda_3^9}{4a_1^2}$.

Because of this superpotential the theory has not a vacuum and exhibits the runaway phenomenon.

5 $N_f = 4$. A smooth quantum moduli space.

In this case because the quarks are not charged under the $U(1)_R$ symmetry, it is not possible to generate a superpotential, which has $U(1)_R$ charge two. Nevertheless the classical theory can be described with the 15 gauge invariants M^{fg} , $B_f \equiv \frac{1}{3!} \epsilon_{ff_1 f_2 f_3} B^{f_1 f_2 f_3}$ and $\mathcal{B} \equiv \mathcal{B}^{1234}$ and one constraint

$$\det M - \mathcal{B}^2 - B_f M^{fg} B_g = 0 \quad (5.1)$$

The necessity for this constraint is easily seen with the help of the naive counting of the complex d.o.f $\sharp(Q) - \sharp(G_2) = 14$ and it can be obtained with the help of the tensor analysis and using the Bose symmetry of the Q s.

If we turn on $W_{\text{tree}} = mM$ and we use the symmetries, we find that

$$\langle M^{fg} \rangle = k_1 \Lambda_4^2 (\det m)^{\frac{1}{4}} (m^{-1})^{fg} \quad \langle B_f \rangle = 0 \quad \langle \mathcal{B} \rangle = k_2 \Lambda_4^4 \quad (5.2)$$

with some unknown constants k_1, k_2 . At the quantum level the constraint eq. (5.1) needs to be modified to

$$\det M - \mathcal{B}^2 - B_f M^{fg} B_g = (k_1^4 - k_2^2) \Lambda_4^8 \quad (5.3)$$

in order to be able to accommodate the v.e.v. eq. (5.2). As in ref. ([4]) this modification is a pure one instanton effect and it has as a consequence the smoothing out of the classical singularity at the origin. One could wonder whether it can happen that the two contributions to the r.h.s. of eq. (5.3) cancel each other. If so there would be an enhanced symmetry at the origin of the moduli space since there would be no smoothing of the singularity. The answer is that it cannot happen as it can be seen decoupling one flavour from the $N_f = 5$ theory where this possibility does not exist. Moreover turning on $W_{\text{tree}} = m_{44} M^{44}$ and decoupling the fourth quark we recover the superpotential for $N_f = 3$ eq. (4.4) when we identify $\Lambda_{3,3}^9 = m_{44}(k_1^4 - k_2^2) \Lambda_4^8$. It should therefore be possible using the \overline{DR} scheme to set $(k_1^4 - k_2^2) = 1$.

Another independent check of this result is obtained by higgsing the theory to $SU(3)$ by $\langle Q^{c1} \rangle = \delta^{c6} a_1$. We get $B^{234} = i\sqrt{2}(\hat{B} - \hat{\tilde{B}})$, $B^{1fg} = ia_1(\hat{M}^{f|g} - \hat{M}^{g|f})$, $\mathcal{B} = a_1\sqrt{2}(\hat{B} + \hat{\tilde{B}})$ which imply the usual $N_f = N_c = 3$ constraint for the $SU(3)$ theory ([4]) when we identify $\Lambda_{3,3}^6 = (k_1^4 - k_2^2) \frac{\Lambda_4^8}{8a_1^2}$.

In order to verify the consistency of previous picture we can check the 't Hooft anomaly matching condition ([11]) at the point $M^{fg} = B_f = 0$ and $\mathcal{B} = \sqrt{-(k_1^4 - k_2^2)} \Lambda_4^4$. At this point the whole global symmetry is unbroken. The microscopic fermions transform in the $14 \times (\underline{1})_1 \oplus 7 \times (\underline{N_f})_{-\frac{4}{N_f}}$ of $SU(N_f = 4) \times U(1)_R$ while the macroscopic ones in the $\left(\frac{N_f(N_f+1)}{2}\right)_{-1} \oplus (\bar{N}_f)_{-1}$. This gives the following identities for the 't Hooft anomalies (we use the coefficient computed in ref. ([11]) p. 153)

$$\begin{aligned} U(1)_R \quad -14 &= -\frac{N_f(N_f+1)}{2} - N_f \\ U(1)_R^3 \quad 14 - \frac{7 \cdot 64}{N_f^2} &= -\frac{N_f(N_f+1)}{2} - N_f \\ SU(N_f)^2 U(1)_R \quad -7d_2(N_f) &= -d_2\left(\frac{N_f(N_f+1)}{2}\right) - d_2(\bar{N}_f) \\ SU(N_f)^3 \quad 7d_3(N_f) &= d_3\left(\frac{N_f(N_f+1)}{2}\right) + d_3(\bar{N}_f) \end{aligned} \quad (5.4)$$

where $d_2(\underline{r})$ and $d_3(\underline{r})$ are respectively the second and third Casimir of the irrep \underline{r} .

6 $N_f = 5$. Confinement without chiral symmetry breaking.

Similarly to the case $N_f = 4$ we find that the classical moduli space is described by the gauge invariants given in eq. (2.4) with 3 constraints

$$\begin{aligned} \epsilon_{ff_1\dots f_4}\epsilon_{gg_1\dots g_4}M^{f_1g_1}\dots M^{f_4g_4} - \mathcal{B}_f\mathcal{B}_g - B_{ff_1}B_{gg_1}M^{f_1g_1} &= 0 \\ M^{fg}\mathcal{B}_g + \frac{1}{8}\epsilon^{ff_1\dots f_4}B_{f_1f_2}B_{f_3f_4} &= 0 \\ B_{f_1g_1}M^{ff_1}M^{gg_1} + \frac{1}{2}\epsilon^{fgf_1f_2f_3}\mathcal{B}_{f_1}B_{f_2f_3} &= 0 \end{aligned} \quad (6.1)$$

where $B_{fg} = \frac{1}{3!}\epsilon_{fgf_1f_2f_3}B^{f_1f_2f_3}$ and $\mathcal{B}_f = \frac{1}{4!}\epsilon_{ff_1f_2f_3f_4}\mathcal{B}^{f_1f_2f_3f_4}$.

Turning on $W_{\text{tree}} = mM$ we find that the symmetries imply that $\langle M^{fg} \rangle = k_1\Lambda_5^2(\det m)^{\frac{1}{4}}(m^{-1})^{fg}$, $\langle B^{f_1f_2f_3} \rangle = 0$ and $\langle \mathcal{B}^{f_1\dots f_4} \rangle = 0$ and we must therefore modify the classical constraint to

$$\epsilon_{ff_1\dots f_4}\epsilon_{gg_1\dots g_4}M^{f_1g_1}\dots M^{f_4g_4} - \mathcal{B}_f\mathcal{B}_g - B_{ff_1}B_{gg_1}M^{f_1g_1} = k_1^4\Lambda_5^7m_{fg} \quad (6.2)$$

Following the line of thought of ref. ([4]) we are lead to conclude that all the gauge invariant fields are required to a complete description of the quantum theory since we can completely fill a neighbourhood of the origin of the quantum moduli space tuning the “external” sources in $W_{\text{tree}} = mM + bB + c\mathcal{B}$, i.e. we find all the possible values of M , B and \mathcal{B} for $m, b, c \neq 0$.

This satisfies a highly non trivial check of consistency: the ’t Hooft anomaly consistency conditions. At the origin of the quantum moduli space the whole global symmetry $SU(5) \times U(1)_R$ is unbroken and the microscopic fields transform in $14 \times (\underline{1})_1 \oplus 7 \times (\underline{5})_{-\frac{4}{5}}$ while the macroscopic ones in the $(\underline{15})_{-\frac{3}{5}} \oplus (\underline{10})_{-\frac{2}{5}} \oplus (\underline{5})_{-\frac{1}{5}}$. The ’t Hooft anomaly conditions, which are satisfied, are

$$\begin{aligned} U(1)_R \quad -14 &= 15 \cdot \left(-\frac{3}{5}\right) + 10 \cdot \left(-\frac{2}{5}\right) + 5 \cdot \left(-\frac{1}{5}\right) \\ U(1)_R^3 \quad 14 - \frac{64 \cdot 7}{N_f^2} &= 15 \cdot \left(-\frac{3}{5}\right)^3 + 10 \cdot \left(-\frac{2}{5}\right)^3 + 5 \cdot \left(-\frac{1}{5}\right)^3 \end{aligned}$$

$$\begin{aligned}
SU(N_f)^2 U(1)_R & -\frac{28}{N_f} d_2(N_f) = d_2(15) \cdot \left(-\frac{3}{5}\right) + d_2(10) \cdot \left(-\frac{2}{5}\right) + d_2(5) \cdot \left(-\frac{1}{5}\right) \\
SU(N_f)^3 & 7d_3(N_f) = d_3(15) + d_3(10) + d_3(5)
\end{aligned} \tag{6.3}$$

Since the theory makes sense at the origin too and we can describe it using all the gauge invariant fields we have, we can expect to be able to find a unique superpotential for the effective low energy Lagrangian. The unique superpotential which respects the symmetries, reproduces the flat directions correctly and yields the previous superpotentials when integrating out quarks is given by

$$W_5 = \frac{1}{\Lambda_5^7} \left(-\det M + \frac{1}{2} B_{f_1 f_2} B_{g_1 g_2} M^{f_1 g_1} M^{f_2 g_2} + \mathcal{B}_f \mathcal{B}_g M^{fg} + \frac{1}{4} \epsilon^{f_1 \dots f_5} \mathcal{B}_{f_1} B_{f_2 f_3} B_{f_4 f_5} \right) \tag{6.4}$$

7 $N_f \geq 6$. The interacting superconformal field theory.

As in the previous case the quantum moduli space has a singularity at the origin but now we cannot satisfy the 't Hooft conditions. The simplest one, the $U(1)_R$ condition, tells that there are not enough fermions with negative $U(1)_R$ charge (for the $N_f = 6$ we get -14 from the microscopic point of view while the macroscopic contribution is -2) but we cannot construct any further operator whose fermions have negative $U(1)_R$ charge. The theory at the origin should be in a non Abelian Coulomb phase since it can be higgsed to a $SU(3)$ theory with $N_f \geq 5$ which is known to be either in such a phase ($N_f \leq 8$) or not asymptotically free ($N_f \geq 9$). Let us therefore examine the dimension of the gauge invariant operators which we have given in eq. (2.4), in the deep infrared supposing that the theory is described by a $N = 1$ superconformal theory ([5]), we get

$$\begin{aligned}
D(M^{fg}) &= \frac{3}{2} D(M^{fg}) = 3 - \frac{12}{N_f} \\
D(B^{f_1 f_2 f_3}) &= \frac{9}{2} - \frac{18}{N_f} \\
D(\mathcal{B}^{f_1 f_2 f_3 f_4}) &= 6 - \frac{24}{N_f}
\end{aligned} \tag{7.1}$$

From the theory of representation of $N = 1$ superconformal theory we know that a unitary representation necessary has $D \geq 1$ for all the operators, this implies $N_f \geq 6$. In particular for $N_f = 6$ if the theory is superconformal in the IR, the mesonic fields M^{fg} is free. The non existence of a gap between the confining phase and the non Abelian Coulomb phase is nothing peculiar of this theory since it happens for $SU(2)$ and $SU(3)$ theories with fundamental matter too.

We could wonder whether it is now necessary to have a dual theory since the very reason calling for a dual theory in the $U(n)$ ([5]), $SO(n)$ ([6]) and $Sp(n)$ ([7]) gauge theories with fundamental matter, i.e. the non existence of a unitary theory in term of the original gauge invariant fields in the deep IR, has disappeared and everything proceeds smoothly from $N_f = 0$ to $N_f = 12$ where the theory ceases to be asymptotically free. The answer is yes, we need to have a dual description because G_2 with $N_f = 6$ can be obtained higgsing $SO(7)$ with $N_f = 7$ flavours of matter in the spinorial irrep $\underline{8}$ and this theory should be in the non-Abelian Coulomb phase since G_2 is but on the other side its meson does not belong to a unitary representation of a superconformal theory.

We can therefore try to construct the dual theory explicitly. To this purpose we want to make use of the commutativity of the diagram

$$\begin{array}{ccc}
(G_2, N_f \cdot \underline{7}) & \xrightarrow{\text{dual}} & (\tilde{G}_2(N_f), N_f \cdot (\sum_i \underline{r_i}) \oplus \text{gauge invariants}) \\
\text{Higgs } \downarrow & & \downarrow \text{decoupling} \\
(SU(3), (N_f - 1) \cdot (\underline{3} + \bar{\underline{3}})) & \xrightarrow{\text{dual}} & (SU(N_f - 4), (N_f - 1) \cdot (\underline{N_f - 4} + \overline{\underline{N_f - 4}}) \oplus \hat{M}^{\hat{f}|\hat{g}}) \\
& & (7.2)
\end{array}$$

where we are assuming that the decoupling is the corresponding phenomenon in the magnetic theory of the higgsing in the electric theory and that there is a unique dual to the $SU(3)$ gauge theory with vector matter. With this assumption we can conclude that $\tilde{G}_2(N_f) = SU(N_f - 4)$ and $\sum_i \underline{r_i} = \underline{N_f - 4} + \overline{\underline{N_f - 4}}$. Now we run into troubles since the global symmetries of the $SU(3)$ and G_2 theories are quite different: the $SU(3)$ symmetries $SU(N_f - 1) \times SU(N_f - 1) \times U(1)_V \times U(1)_R$ are not a subgroup of the G_2 symmetries $SU(N_f) \times U(1)_R$. We can partially solve the problem with the superpotential $\tilde{W} = M^{fg} q_f \bar{q}_g$ which breaks $SU(N_f) \times SU(N_f)$ to $SU(N_f)$ since M^{fg} is symmetric. We are then obliged to introduce B^{fgh} as elementary field of the dual since we want to identify B^{1fg} with the antisymmetric part of $\hat{M}^{\hat{f}|\hat{g}}$. Another reason to introduce B as an elementary field is that it is

not possible to construct any operator using the dual quarks with the same $U(1)_R$ charge of B . But we are now faced with the problem of eliminating a linear combination of the dual baryon and dual antibaryon from the chiral ring since we have only one baryon with four indeces in the electric theory. Associated with this problem there is also the way of eliminating the global exceeding $U(1)_V$ symmetry. We were not able to solve this problem but we hope to return on it in another paper.

Appendix A: The group G_2

The group G_2 is the group to which $SO(7)$ is spontaneously broken by a spinor S in the $\underline{8}$ of $SO(7)$. G_2 has therefore dimension 14 and rank 2 and all its irreps are real.

The branching rules of $SO(7)$ in G_2 are

$$\begin{aligned}\underline{7} &\rightarrow \underline{7} \\ \underline{21} &\rightarrow \underline{7} + \underline{14}\end{aligned}\tag{A.1}$$

and both $\underline{7}$ and $\underline{14}$ are elementary irreps of G_2 but only $\underline{7}$ is a simple irrep.

In particular $\underline{7} \times \underline{7} = (\underline{1} + \underline{27})_{\text{symm}} + (\underline{7} + \underline{14})_{\text{antisymm}}$ therefore the confining phase and the Higgs phase are indistinguishable since there is no Wilson loop which cannot be shielded by quarks.

G_2 is characterised by two invariants ([12])

1. δ_{ab}
2. Γ_{abc} which is totally antisymmetric

where $a, b \dots = 1, \dots 7$ and

$$\Gamma_{i..}^{ln} \Gamma_{j..}^{k.} + \Gamma_{j..}^{ln} \Gamma_{i..}^{k.} = 2\delta_{ij} \delta^{lk} - 2\delta_{\{i}^l \delta_{j\}}^k = 2\delta_{ij} \delta^{lk} - \delta_i^l \delta_j^k - \delta_i^k \delta_j^l \tag{A.2}$$

Moreover Γ_{abc} can be interpreted as $\Gamma_{abc} = \langle \bar{S} \gamma_{abc} S \rangle$ in view of the embedding $G_2 \subset SO(7)$ described at the beginning.

The previous formula eq. (A.2) allows the complete reduction of the products of three or more Γ :

$$\Gamma^{mns} \Gamma_{sti} \Gamma_{jk}^t = 2\delta_{i[j} \Gamma_{k]}^{mn} - 6\delta_{[i}^{[m} \Gamma_{jk]}^{.n]} \tag{A.3}$$

In order to be able to find the constraints among the gauge invariant fields we need the following decompositions into irreducible parts

$$\begin{aligned}
T_{[c_1 c_2 c_3]} &= \frac{1}{7 \cdot 3!} \Gamma_{c_1 c_2 c_3} T + \frac{1}{24} \Gamma_{[c_1 c_2}^p \Gamma_{c_3] k p} T^k + \frac{3}{4} \left(\Gamma_{[c_1 c_2}^{\{k} \delta_{c_3] l}^{\}} - \frac{1}{7} \Gamma_{c_1 c_2 c_3} g^{kl} \right) T_{\{kl\}} \\
&= -\frac{1}{12} \Gamma_{c_1 c_2 c_3} (\Gamma^{d_1 d_2 d_3} T_{d_1 d_2 d_3}) + \frac{1}{2} (\Gamma^2)_{[c_1 c_2}^{kl} T_{c_3] kl} + \frac{1}{4} \Gamma_{d_1 [c_1 c_2} \Gamma_{c_3] d_2 d_3} T^{d_1 d_2 d_3}
\end{aligned} \tag{A.4}$$

where $(\Gamma^2)_{c_1 c_2}^{kl} = \Gamma^{skl} \Gamma_{s c_1 c_2}$ and T , T_k and $T_{\{kl\}}$ are the 1, 7 and 27 irreducible parts given by

$$\begin{aligned}
T &= \Gamma^{c_1 c_2 c_3} T_{c_1 c_2 c_3} \\
T_k &= \Gamma_{[c_1 c_2}^p \Gamma_{c_3] k p} T^{c_1 c_2 c_3} \\
T_{\{kl\}} &= \left(\Gamma_{c_1 c_2 \{k} \delta_{l\} c_3\} - \frac{1}{7} g_{kl} \Gamma_{c_1 c_2 c_3} \right) T^{c_1 c_2 c_3}
\end{aligned} \tag{A.5}$$

Moreover we need

$$T_{[c_1 \dots c_4]} = \frac{1}{7 \cdot 4!} \Gamma_{[c_1 c_2}^p \Gamma_{c_3 c_4] p} T + \frac{1}{6} \Gamma_{[c_1 c_2 c_3} \delta_{c_4] k} T^k + \frac{3}{16} \left(\Gamma_{k [c_1 c_2} \Gamma_{c_3 c_4] l} - \frac{1}{7} \delta_{kl} \Gamma_{p [c_1 c_2} \Gamma_{c_3 c_4] p} \right) T_{\{kl\}} \tag{A.6}$$

where T , T_k and $T_{\{kl\}}$ are the 1, 7 and 27 irreducible parts given by

$$\begin{aligned}
T &= \Gamma_{p [c_1 c_2} \Gamma_{c_3 c_4] p} T^{[c_1 \dots c_4]} \\
T_k &= \Gamma^{c_1 c_2 c_3} T_{[c_1 c_2 c_3 k]} \\
T_{\{kl\}} &= \left(\Gamma_{k [c_1 c_2} \Gamma_{c_3 c_4] l} - \frac{1}{7} g_{kl} \Gamma_{[c_1 c_2}^p \Gamma_{c_3 c_4] p} \right) T_{[c_1 \dots c_4]}
\end{aligned} \tag{A.7}$$

We also need

$$T_{[c_1 \dots c_5]} = \frac{5}{72} \Gamma_{[c_1 c_2}^p \Gamma_{c_3 c_4 c_5] p} T_p + \frac{5}{27} \left(\frac{1}{2} \Gamma_{[c_1 c_2 c_3} \Gamma_{c_4 c_5] p} \Gamma^{pkl} + \Gamma_{[c_1 c_2 c_3} \Gamma_{c_4 \dots}^{kp} \Gamma_{c_5] \dots}^{l.} \right) T_{[kl]}^{(14)} \tag{A.8}$$

where T_k and $T_{[kl]}^{(14)}$ are the 1 and 14 irreducible parts given by

$$\begin{aligned}
T_k &= \Gamma^{c_1 c_2 c_3} \Gamma_{c_4 c_5 k} T^{[c_1 c_2 c_3 c_4 c_5]} \\
T_{[kl]}^{(14)} &= \left(\frac{1}{2} \Gamma_{[c_1 c_2 c_3} \Gamma_{c_4 c_5] p} \Gamma^{pkl} + \Gamma_{[c_1 c_2 c_3} \Gamma_{c_4 \dots}^{kp} \Gamma_{c_5] \dots}^{l.} \right) T^{[c_1 c_2 c_3 c_4 c_5]}
\end{aligned} \tag{A.9}$$

Appendix B: Explicit representation of G_2

We took the explicit representation from the first reference of ([12]) which we checked and corrected.

$$g_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} \frac{1}{4\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{4\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{4\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2\sqrt{3}} \end{pmatrix} \quad H_2 = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0 & \frac{1}{2\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2\sqrt{6}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
E_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{-1}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2\sqrt{6}} & 0 & 0 & 0 \end{pmatrix} & E_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix} \\
E_{-\alpha} &= E_{\alpha}^{\dagger} \\
tr(E_{-\alpha}E_{\alpha}) &= tr(H_i^2) = \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
N_{2,6}^4 &= N_{4,-6}^2 = N_{-2,4}^6 = N_{2,-1}^3 = N_{3,1}^2 = N_{-2,3}^{-1} \\
&= N_{5,6}^1 = N_{1,6}^5 = N_{-1,5}^6 = N_{-4,3}^{-5} = N_{4,-5}^3 = N_{-5,-3}^{-4} = \frac{1}{2\sqrt{2}} \\
N_{1,5}^3 &= N_{3,-5}^1 = N_{-1,3}^5 = \frac{1}{\sqrt{6}} \\
\alpha_1 &= (\frac{1}{2\sqrt{3}}, 0) \quad \alpha_3 = (\frac{1}{4\sqrt{3}}, \frac{1}{4}) \quad \alpha_5 = (-\frac{1}{4\sqrt{3}}, \frac{1}{4}) \\
\alpha_2 &= (\frac{\sqrt{3}}{4}, \frac{1}{4}) \quad \alpha_4 = (0, \frac{1}{2}) \quad \alpha_6 = (-\frac{\sqrt{3}}{4}, \frac{1}{4}) \\
\Gamma^{137} &= \Gamma^{245} = i\sqrt{2} \quad \Gamma^{146} = \Gamma^{236} = -\Gamma^{567} = -i
\end{aligned}$$

Appendix C: $SU(3) \subset G_2$

We will consider the simplest way of embedding (higgsing) $SU(3)$ into G_2 . This is achieved by turning on the v.e.v $\langle Q^c \rangle = a\delta_6^c$. In this way the $SU(3)$ is generated by $\{H_1, H_2, E_{\pm 2}, E_{\pm 4}, E_{\pm 6}\}$. Using the matrices of the previous appendix is easy to realize that

$$\begin{aligned}
\{Q^1, Q^7, Q^3\} &\rightarrow q^c \\
\{Q_1, Q_7, Q_3\} &\equiv \{Q^4, -Q^5, -Q^2\} \rightarrow \bar{q}_c \\
\{Q^6\} &\rightarrow s
\end{aligned} \tag{C.1}$$

where q^c , \bar{q}_c and s transform respectively as $\underline{3}$, $\bar{\underline{3}}$ and $\underline{1}$ of $SU(3)$.

Using the previous formulae eq.s (C.1) we can decompose the gauge invariants operators (2.4), we get ($f, g \geq 2$)

$$\begin{aligned}
M^{fg} &= \hat{M}^{fg} + \hat{M}^{gf} \\
B^{1fg} &= ia(\hat{M}^{fg} - \hat{M}^{gf}) \\
B^{f_1 f_2 f_3} &= i\sqrt{2}(\hat{B}^{f_1 f_2 f_3} - \hat{\bar{B}}^{f_1 f_2 f_3}) \\
\mathcal{B}^{1f_1 f_2 f_3} &= \sqrt{2}a(\hat{B}^{f_1 f_2 f_3} + \hat{\bar{B}}^{f_1 f_2 f_3}) \\
\mathcal{B}^{f_1 f_2 f_3 f_4} &= -4\hat{M}^{[f_1 f_2} \hat{M}^{f_3 f_4]} + 4\sqrt{2}(s^{[f_1} \hat{B}^{f_2 f_3 f_4]} + s^{[f_1} \hat{\bar{B}}^{f_2 f_3 f_4]}) \quad (C.2)
\end{aligned}$$

where

$$\begin{aligned}
\hat{M}^{fg} &= q^{cf} \bar{q}_c^g + \frac{1}{2} s^f s^g \\
\hat{B}^{f_1 f_2 f_3} &= \epsilon_{c_1 c_2 c_3} q^{c_1 f_1} q^{c_2 f_2} q^{c_3 f_3} + \frac{3\sqrt{2}}{2} s^{[f_1} \hat{M}^{f_2 f_3]} \\
\hat{\bar{B}}^{f_1 f_2 f_3} &= \epsilon_{c_1 c_2 c_3} \bar{q}^{c_1 f_1} \bar{q}^{c_2 f_2} \bar{q}^{c_3 f_3} - \frac{3\sqrt{2}}{2} s^{[f_1} \hat{M}^{f_2 f_3]} \quad (C.3)
\end{aligned}$$

where the sign in the last two equations is the right one compatible with the pseudo conjugation exchanging q with \bar{q} .

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